ANALYSIS OF EXPERIMENTS INVOLVING RANKINGS IN TRIAD COMPARISONS

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1. INTRODUCTION

Ranking methods are generally used where quantitative observations are not obtained easily. Moreover, because of its simplicity, sometimes it is used in order to reduce the labour of computation or to get a rapid result. Thus analysis of experiments based on ranking items has received considerable attention.

Various authors have proposed different methods for the rank analysis. The analysis of paired comparisons has attracted the attention of many authors including Thurstone [10], Kendall and Babington Smith [7], Bradley and Terry [1]. The analysis of experiments involving ranking in triple comparisons has been developed by Pendergrass and Bradley [8]. Here we shall develop a method of analysis of rankings in triad comparisons. A model is postulated and estimates of the treatment parameters are obtained. Suitability of the model and combination of results from different experiments are discussed.

2. MATHEMATICAL MODEL

Let us consider v treatments in an experiment involving triple comparisions. It is supposed that the treatment $T_1, T_2 \ldots, T_v$ have true ratings (or preferences), $\pi_1, \pi_2, \ldots, \pi_v$ on a particular subjective continuum which satisfies the following conditions.

$$(i)\ \pi_i \geqslant 0 \qquad \ldots(2.1)$$

$$(ii) \sum_{i=1}^{v} \pi_i = 1 \qquad \ldots(2.2)$$

The total number of triplets formed out of v treatments is $\binom{v}{3}$. The number of each of $\binom{v}{3}$ triplets will be ranked in order of acceptability. In a triplet the best treatment will be given rank 1, the second rank 2 and the third rank 3. When treatment i appears with treatments j and k in a block, we shall indicate $p\ (T_i > T_j > T_k)$ as the probability that treatment i obtains top rating treatment j the middle and treatment k the last.

We shall take the model:

$$\pi_{ijk} = P\ (T_i > T_j > T_k) = \pi_i^4\ \pi_j^2 / \triangle_{ijk}, \ldots \qquad \ldots(2.3)$$

where

$$\triangle_{ijk} = \pi_i^4 \left(\pi_j^2 + \pi_k^2\right) + \pi_j^4 \left(\pi_i^2 + \pi_k^2\right) + \pi_k^4 \left(\pi_i^2 + \pi_j^2\right)$$

3. THE LIKELIHOOD FUNCTION

The method of maximum likelihood will be used to obtain estimators $p_1, p_2, \ldots, p_v$ of parameters $\pi_1, \pi_2, \ldots, \pi_v$. We may obtain the likelihood function assuming the probability idependence for different triplets and for different repetitions. The ranks of T_i, T_j and T_k in the m-th comparison will be written by $r_{im,\,jk}$; $r_{jm,\,ik}$ and $r_{km,\,ij}$ respectively: $m=1, \ldots, n$. Tied ranks are not permitted in the model. The probability of a specified ranking in the m-th repetition is given by

$$\left(\pi_i^2\right)^{3-r_{im,\,jk}} \left(\pi_j^2\right)^{3-r_{jm,\,ik}} \left(\pi_k^2\right)^{3-r_{km,\,ij}} (\triangle_{ijk})^{-1} \ldots, \qquad \ldots(3.1)$$

For, if the treatment i obtains top rank, the treatment j the second and k the third *i.e.*

$$r_{im,\,jk} = 1;$$
$$r_{jm,\,ik} = 2;$$
$$r_{km,\,ij} = 3,$$

then the expression (3.1) reduces to $\pi_i^4 \quad \pi_j^2\ \triangle_{ijk}^{-1}$. And if

$$r_{im,\,jk} = 1,$$
$$r_{jm,\,ik} = 3,$$
$$r_{km,\,ij} = 2$$

the expression becomes: $\pi_i^4 \quad \pi_k^2 \quad \Delta_{ijk}^{-1}$ and so on. When we multiply the appropriate expression for all triplets within a repetition and for all n repetitions, we obtain the likelihood function as :

$$L=\frac{\prod_{i=1}^{v}\left(\pi_i^2\right)^{3\frac{n}{2}(v-1)(v-2)-\sum_{j<k}^{v}\sum_{m=l}^{n} r_{im,jk}}}{\prod_{i<j<k}^{v}(\Delta_{ijk})^n}$$

$$=\frac{\prod_{i=1}^{v}\pi_i^{3n(v-1)(v-2)-2\sum_{j<k}^{v}\sum_{m<l}^{n} r_{im,jk}}}{\prod_{i<j<k}^{v}(\Delta_{ijk})^n} \qquad \ldots(3.2)$$

4. LIKELIHOOD RATIO TESTS AND ESTIMATION

We can test the significance of the equality of treatment effects. Consider :

$$Ho: \pi_i=\frac{1}{v} \qquad \text{for all } i;\ i=1, \ldots, v$$

$$H1: \pi_i \neq \pi_j \qquad \text{for some } i\neq j;\ i, j=1, \ldots, v$$

The maximum likelihood estimator $P_1, P_2, \ldots, P_v$ are obtained by maximising $\log L$ with respect to $\pi_1, \ldots, \pi_v$ subject to the condition that $\sum_{i=1}^{v}\pi_i=1$. The resulting normal equations are :

$$\frac{a_i}{P_i}=n\sum_{j<k}\left\{\left[4P_i^3\left(P_j^2+P_k^2\right)+2P_i\left(P_j^4+P_k^4\right)\right]\Big/D_{ijk}\right\} \qquad \ldots(4.1)$$

$i=1, \ldots, v$

where

$$a_i = 3n\,(v-1)\,(v-2) - 2 \sum_{j<k}^{v} \sum_{m=1}^{n} r_{im\cdot jk}$$

$$D_{ijk} = P_i^4\left(P_j^2 + P_k^2\right) + P_j^4\left(P_i^2 + P_k^2\right) + P_k^4\left(P_i^2 + P_j^2\right).$$

Solution of the equations (4.1) will give the values of $P_1, \ldots, P_v$. The normal equations are solved by iterative methods.

The π_i's are completely specified under H_o and have values P_i's which can be obtained by the above procedure.

The likelihood function given by (3.2) is used to obtain the likelihood ratio λ.

$$\lambda = \frac{L\,(a,\,\pi)\,H_0}{L\,(a,\,\pi)\,H_l}$$

$$= \frac{\prod_{i=1}^{v}\left(\frac{1}{v}\right)^{a_i}}{\prod_{i<j<k}^{v}\left(\Delta_{ijk}^{n}\right)_{H0}} \times \frac{\prod_{i<j<k}^{v}\left(\Delta_{ijk}^{n}\right)_{H_l}}{\prod_{i=1}^{v}\pi_i^{a_i}}$$

Then Z can be obtained as it is given by

$$Z = -2 \log_e \lambda.$$

$$Z = 2n\binom{v}{2}\log_e 6 + 2\sum_{i} a_i \log_e \pi_i - 2n \sum_{i<j<k} \Delta_{ijk} \quad \ldots(4.3)$$

For large n, Z may be taken to have the chi-square distribution with $(v-1)$ degree of freedom (Wilks) under the null hypothesis H_0.

Small-Sample tables for the distribution of Z given H_0 may be developed but these would be extremely voluminous. The procedure for developing such tables is similar to that used by Bradley and Terry [1] and Rai and Sadasivan [9]. An example of such tables is given in Table 1, where Z_0 indicates the values of Z for specified sets of sums of ranks in the table.

TABLE 1

Distribution of $Z=-2 \log_e \lambda$ for $\nu=4$, $n=1$

Rank Sums				*Estimates of π_i*				*Distribution*	
Σr_1	Σr_2	Σr_3	Σr_4	P_1	P_2	P_3	P_4	Z_0	$P(Z \geqslant Z_0)$
3	5	7	1	1				14.3341	.0185
3	5	8	8	1				14.3341	.0370
3	6	6	9	1				14.3341	.0556
3	6	7	8	1				14.3341	.1296
3	7	7	7	1				14.3341	.1482
4	4	7	9	.4546	.4546	.0908	—	10.8895	.1667
4	4	8	8	.4284	.4284	.0716	.0716	9.2356	.1852
4	5	6	9	.4490	.3195	.2315	—	7.6572	.2593
5	5	5	9	.3333	.3333	.3333	—	6.6184	.2778
4	5	7	8	.3848	.3090	.1906	.1156	4.2872	.3889
4	6	6	8	.3940	.2505	.2505	.1050	3.9562	.4630
4	6	7	7	.3829	.2335	.1918	.1918	2.4126	.5741
5	5	6	8	.3134	.3134	.2551	.1181	2.3677	.5926
5	5	7	7	.3159	.3159	.1841	.1841	1.5086	.7038
5	6	6	7	.3124	.2442	.2442	.1994	0.7180	.7964
6	6	6	6	.2500	.2500	.2500	.2500	0.0000	1.0000

5. Combination of Results

In certain situations, experiments may be performed in groups of repetitions of sizes n_u $(u=1, \ldots, g)$ with $\sum_{n=1}^{g} n_u = n$. These groups may be judges at different times or under different circumstances. Distinct treatment parameters $\pi_{1u}, \ldots, \pi_{vu}$ may exist for such group. The failure of the treatment parameters to be the same for each group; represents a group-into-treatment intreraction or lack of agreements. There are different methods for testing the significance depending upon the specification of the alternative hypotheses. We can develop to detect group-treatment interactions.

$$\text{Consider } H_o : \pi_{iu} = \frac{1}{v} \text{ for all } i \text{ and } u$$

$$H_a : \pi_{iu} \neq \frac{1}{v} \text{ for some } i \text{ and } u$$

If λ_c is the likelihood ratio in this situation,

$$\text{then } Z_c = -2 \log_e \lambda c = \sum_{u=l}^{g} Z_u ;$$

where Z_u is the Z of (4.3) computed for the u-th group. For large n_u, Z_e has the χ^2 distribution with g (v-1) degree of freedom under H_o. This test may be designated as the combined test of treatment equality or for main effects.

Consider $H_o : \pi_{iu} = \pi_i ;\ i = l, \ldots, v ;\ u = l, \ldots, g.$

and $H_a : \pi_{iu} \neq \pi_i$ for some i and u.

The likelihood ratio test of this case depends on $Z_e - Z$ and has the χ^2 distribution with $(g-l)\ (v-l)$ degree of freedom.

The large sample test procedure are summarised in Table 2.

TABLE 2

Large Sample Test of Significance

Test	*Hypothesis*	*Statistic*	*Limiting distribution*
Main effect (No Interaction)	$H_o : \pi_i = \frac{1}{v}$	Z	$\chi^2_{(v-1)}$
Interaction	$H_o : \pi_{iu} = \pi_i$ $H_a : \pi_{iu} \neq \pi_i$	$Z_c - Z$	$\chi^2_{(g-1)(v-1)}$
Main effect (Interaction)	$H_o : \pi_{iu} = \frac{1}{v}$ $H_a : \pi_{iu} \neq \frac{1}{v}$	Z_c	$\chi^2_{g(v-1)}$

6. Appropriateness of the Model

It is desirable that means be available to test the appropriateness of the model on which the method is based. The mathematical model for triple comparisons is postulated in such a way that it is mathematically workable and easy to apply and interpret. The existence of the non-negative parameters is assumed (2.1). There are six possible rankings of T_i, T_j and T_k in each triplet. The six parameters $\pi_{ijk}, \ldots\ldots, \pi_{kji}$ sum to unity for each triplet and their maximum likelihood estimators are $f_{ijk}/n, \ldots, f_{kji}/n$ for the n comparisons on the triplet where f_{ijk} is the number of times the ranking 1, 2 and 3 for T_i, T_j and T_k respectively occurs in the n triplets.

$$p_{ijk} = f_{ijk}/n \; ; \; i \neq j \neq k \; ; \; i, j, k = 1, \ldots\ldots, v$$

The basic model for triple comparisons implies that

$$H_o : \pi_{ijk} = \pi_i^4 \, \pi_j^2 \Big/ \Delta_{ijk} \text{ as in } (2.3)$$

and $$H_a : \pi_{ijk} \neq \pi_i^4 \, \pi_j^2 \Big/ \Delta_{ijk} \text{ for some } i, j, k.$$

The general likelihood function for triple comparison is

$$\bar{L}(\pi_{ijk}) = C \prod_{i<j<k} \prod_{(i,j,k)} \pi_{ijk}^{\,f_{ijk}} \; ; \qquad \ldots(6.1)$$

Let us define f'_{ijk} as the expected frequency corresponding to the observed frequency f_{ijk}, then the estimates of the expected frequency under H_o is given by

$$f'_{ijk} = n \, p_i^4 \, p_j^2 \Big/ D_{ijk} \; ; \qquad \ldots(6.2)$$

The likelihood ratio statistic depends on $\bar{L}\ (p_{ij}/H_o)$ and $\bar{L}\ (p_{ij}/H_1)$ where $\bar{L}$ is as defined in (6.1). Using f_{ijk} and f'_{ijk}, we can have

$$\bar{L}(p_{ijk}/H_o) = C \prod_{i<j<k} \prod_{(i,j,k)} {f'_{ijk}}^{\,f_{ijk}} \; ; \qquad \ldots(6.3)$$

$$\bar{L}(p_{ijk}/H_1) = C \prod_{i<j<k} \prod_{(i,j,k)} f_{ijk}^{\,f_{ijk}} \; ; \qquad \ldots(6.4)$$

The likelihood ratio statistic for testing H_o is given in terms of frequencies by

$$Z=-2\log_e \lambda=2 \sum_{i<j<k}^{\nu} \sum_{(i,j,k)} f_{ijk} \log_e \left(f_{ijk}/f'_{ijk} \right); \qquad \ldots(6.5)$$

For large n, the statistic has a χ^2—distribution with

$$\left[5 \binom{\nu}{3} - (\nu-1) \right]$$

degree of freedom.

The usual procedure for tests of goodness of fit is to form chi-square by taking sums of terms of the form

$$\left(f_{ijk}-f'_{ijk}\right)^2 / f'_{ijk}.$$

This can be derived from (6.5) as given below :

Let $f_{ijk}/f'_{ijk}=1+e_{ijk}$; where

e_{ijk} may take either positive or negative values.

Then

$$Z=-2\log_e \lambda=2 \sum_{i<j<k}^{\nu} \sum_{(i,j,k)} f'_{ijk} \, (1+e_{ijk}) \log(1+e_{ijk})$$

Expanding the logrithmic series in powers of e_{ijk} and ignoring the terms greater than e^2_{ijk}, we have

$$Z \approx 2 \sum_{i<j<k}^{\nu} \sum_{(i,j,k)} f'_{ijk}(1+e_{ijk})\left(e_{ijk}-\frac{e^2_{ijk}}{2}\right); \qquad \ldots(6.6)$$

Error committed by taking approximation will not be large if $|e_{ijk}|$ is small. We notice that $\sum f'_{ijk} \, e_{ijk}=0$ and equation (6.6) takes the form

$$Z \approx \sum \sum f'_{ijk} \, e^2_{ijk}$$

By putting the values of e_{ijk},

$$Z \approx \sum \sum \left(f_{ijk} - f'_{ijk}\right)^2 / f'_{ijk}$$

7. Distribution of The Estimators

Having obtained maximum likelihood estimators $p_1, p_2, \ldots, p_v$ of the parameters $\pi_1, \pi_2, \ldots, \pi_v$ we gave here the large sample distribution of these estimators and their asymptotic variances and covariances.

Let us define X_i as the number of times treatment i is best preferred to others. The likelihood function in terms of X_i is given by

$$f(X, \pi) = \prod_{i=1}^{v} \pi_i^{2X_i} \Big/ \prod_{i<j<k} \triangle_{ijk} \; ; \qquad \ldots(7.1)$$

where X is the vector of $X_1, X_2, \ldots, X_v$

and π is the vector of $\pi_1, \pi_2, \ldots, \pi_v$

If $X_{i(m)}$ is the observation on X_i in the m-th of n repetitions and the association with $a_i (= 3n \; (v-1) \; (v-2) - 2 \; \Sigma \; \Sigma \; r_{im,\, jk})$ is

$$\sum_{m=1}^{n} X_{i(m)} = a_i \; ; \; (i=1, \ldots v) \qquad \ldots(7.2)$$

Then, the likelihood function can be written as

$$L = f(x, \pi) = \prod_{m=1}^{n} f(X_{(m)}, \pi) \qquad \ldots(7.3)$$

Let x_{ijk} be an indicator variate with the value unity if treatment i ranks over j and K zero other-wise.

then $$X_i = \sum_{j<k} X_{ijk} \; ;$$

x_{ijk} is a multi normal variate with expectation

$$E(X_{ijk})=\pi_{ijk}$$

$$=\pi_i^4\,\pi_j^2\,/\triangle_{ijk}$$

and variance and covariance

$$V(X_{ijk})=\pi_{ijk}(1-\pi_{ijk})$$

$$=\frac{\pi_i^4\,\pi_j^2}{\triangle_{ijk}}\times\frac{\triangle_{ijk}-\pi_i^4\,\pi_j^2}{\triangle_{ijk}}$$

$$=\pi_i^4\,\pi_j^2\,\triangle_{ijk}^{-1}-\pi_i^8\,\pi_j^4\,\triangle_{ijk}^{-2}$$

$$\text{Cov}(X_{ijk},\,X_{jki})=-\pi_{ijk}\,\pi_{jki}$$

$$=-\pi_i^4\,\pi_j^6\,\pi_k^2\,\triangle_{ijk}^{-2}$$

Then, the variates x_{ijk} making up the sum x_i are independent in probability and follows that,

$$E(x_i)=\pi_i^4\sum_{j\neq k}\pi_j^2\,\triangle_{ijk}^{-1}\,; \qquad \ldots(7.4)$$

$$V(x_i)=\pi_i^4\sum_{j<k}\left(\pi_i^2+\pi_k^2\right)\triangle_{ijk}^{-1}-\pi_i^8$$

$$\sum_{j<k}\left(\pi_j^4+\pi_k^4+2\pi_j^2\,\pi_k^2\right)\triangle_{ijk}^{-2}\,;$$

$$i,j,k=1,\ldots,v\,; \qquad (7.5)$$

$$\text{Cov}(x_i,\,x_j)=-\pi_i^4\,\pi_j^4\left[\sum_{k\neq i,j}\triangle_{ijk}^{-2}\left(\pi_i^2\,\pi_j^2+\pi_i^2\,\pi_k^2+\pi_j^2\,\pi_k^2+\pi_k^4\right)\right];$$

$$i\neq j\,;\, i,j,k=1,\ldots,v\,; \qquad (7.6)$$

It is convenient to define :

$$\lambda_{ii}=\pi_i^2\left[\sum_{j<k}\left(\pi_j^2+\pi_k^2\right)\triangle_{ijk}^{-1}-\pi_i^4\right.$$

$$\sum_{j<k} \left(\pi_j^4 + \pi_k^4 + 2\pi_j^2 \pi_k^2 \right)\Delta_{ijk}^{-2}\Bigg];\ i=1, \ldots, v$$

$$\lambda_{ij} = -\pi_i^3 \pi_j^3 \Bigg[\sum_{k \neq i, j} \Delta_{ijk}^{-2} \left(\pi_i^2 \pi_j^2 + \pi_i^2 \pi_k^2 + \pi_j^2 \pi_k^2 \pi + {}_k^4 \right) \Bigg];$$

$$i \neq j\ ;\ i, j=1, \ldots, v \quad (7.7)$$

Then, we have

$$V(X_i) = \pi_i^2 \lambda_{ii} \qquad \ldots(7.8)$$

$$\text{Cov}(X_i, X_j) = \pi_i \pi_j \lambda_{ij}$$

Under general condition, p_i is a consistant estimator of π_i on $f(x_{(m)}, \pi)$ [such as given by Chanda (6)]. The parameters π_i ; $(i=1, \ldots, v)$ are subject to the restriction $\Sigma \pi_i = 1$ and are not independent. If we regard

$$\pi_v = 1 - \sum_{i=1}^{v-1} \pi_i$$

as a function of the first $(v-1)$ parameters and p_v similarly as a function of the first $v-1$ estimators ; $\sqrt{n}\ (p_1 - \pi_1), \ldots, \sqrt{n}\ (p_{v-1} - \pi_{v-1})$ have a joint limiting normal distribution with zero mean and dispersion matrix given by (σ_{ij})

$$(\sigma_{ij}) = \left(\lambda'_{ij} \right)^{-1}., \text{ where}$$

$$\lambda'_{ij} = E\left[\frac{\partial \log_e f(x_m, \pi)}{\partial \pi_i} \ \frac{\partial \log_e f(x_m, \pi)}{\partial \pi_i} \right]; \qquad \ldots(7.9)$$

where

$$\frac{\partial \log_e f(x_m, \pi)}{\partial \pi_i} = \frac{x_i}{\pi_i} - \frac{x_v}{\pi_v} - \sum_{j<k} \Bigg\{ \Bigg[4\pi_i^3 \left(\pi_j^2 + \pi_k^2 \right)$$

$$+ 2\pi_i \left(\pi_j^4 + \pi_k^4 \right) \Bigg] \Big/ D_{ijk} \Bigg\}$$

$$+ \sum_{j<k} \Bigg\{ \Bigg[4\pi_v^3 \left(\pi_j^2 + \pi_k^2 \right) + 2\pi_v \left(\pi_j^4 + \pi_k^4 \right) \Bigg] \Big/ D_{ijk} \Bigg\};$$

$$I=1, \ldots, (v-1).$$

and from (7.4), that

$$\lambda'_{jj} = \text{Cov}\left[\left(\frac{x_i}{\pi_i} - \frac{x_v}{\pi_v} \right) \left(\frac{x_j}{\pi_j} - \frac{x_v}{\pi_v} \right) \right];$$

$i=j=1, \ldots, (v-1)$. From (7.8), we get

$$\lambda'_{ij} = \lambda_{ij} - \lambda_{iv} - \lambda_{jv} + \lambda_{vv}, \ldots, \quad \ldots(7.10)$$

The matrix $\left[\lambda'_{ij} \right]$ is non-negative definite, since it is a dispersion matrix and it is positive definite since $x_1, \ldots, x_{v-1}$ and hence $\partial \log_e f/\partial\pi_1, \ldots, \partial \log_e f/\partial\pi_{v-1}$ are free of linear restrictions. Thus we conclude that $\sqrt{n}\,(p_1-\pi_1), \ldots, \sqrt{n}\,(p_{v-1}-\pi_{v-1})$ have the multivariate normal distribution with zero means and dispersion matrix $\left(\lambda'_{ij} \right)^{-1}$. This result holds true for large samples.

VARIANCES AND COVARIANCES OF ESTIMATORS

We note that $v \times v$ matrix (λ_{ij}) is singular in view of the definitions (7.7)

$$\pi_i^2 \lambda_{ii} + \sum_j \pi_j^2 \lambda_{ij} = 0$$

We may resort to the similar algebric manipulations as used by Bradley [5] for paired comparison. If element σ_{ij} of $(\lambda_{ij})^{-1}$ are defined, the result is that

$$\sigma_{ij} = \frac{\text{cofactor of } \lambda_{ij} \text{ in} \begin{bmatrix} (\lambda_{ij}) & e' \\ e & o \end{bmatrix}}{\begin{vmatrix} (\lambda_{ij}) & e' \\ e & o \end{vmatrix}}; \quad \ldots(7.11)$$

$$i, j = 1, \ldots, (v-1)$$

Where e denotes the v element row vector of unit elements, and e' is the transpose of e. It follows from symmetry that (7.11) holds for $i, j = 1, \ldots v$ and we may define a v-square matrix $\Sigma = (\sigma_{ij})$ that is singular with rank $(v-1)$. It follows that $\sqrt{n}\,(p_1-\pi_1), \ldots, \sqrt{n}\,(p_v-\pi_v)$ have asymptotically with n, the singular v-variate normal distribution with zero means and dispersion matrix Σ.

In general we may take $p_1, \ldots, p_v$ as jointly normally distributed with means $\pi_1, \ldots, \pi_v$ and dispersion matrix $(\sigma_{ij})/n$ for large samples.

Usually estimated variances and covariances are required. We may define $\hat{\lambda}_{ij}$ $(i, j = 1, \ldots, v)$ to be the same functions of $p_1, \ldots, p_v$ as λ_{ij} are of $\pi_1 \ldots, \pi_v$ as defined in (7.7), since maximum likelihood estimates are consistant. Thus $(\hat{\sigma}_{ji})$ is the dispersion matrix which is the same function of $\hat{\lambda}_{ij}$ as (σ_{ij}) is of λ_{ij}

8. Discussion and Summary

In this study we have developed a method of analysis of experiment involving triad comparisions which permits tests of hypothesis of general class and estimation of treatment ratings or perference. In the null hypothesis we assume that the treatment ratings are equal whereas the alternative hypothesis make no assumption regarding the equality of the treatment ratings. The probability of the sums of ranks $p(r_i<r_j<r_k)$ involves three paired comparison consisting of pairs of treatment (T_i, T_j) ; (T_i, T_k) and (T_j, T_k). These comparisons should be consistant in which $r_i<r_j$; $r_i<r_k$ and $r_j<r_k$.

The approach here may also be used for the generalisation of ranking in Block of size greater than 3. In subjective testing involving taste or odours, paired or triple comparions will satisfy most of the requirements of the experimenter.

Formulae for the variances and covariances of estimates of treatment ratings $\pi_1, ..., \pi_v$ have been obtained. A test for the appropriateness of the model is given. The method of combining the results is also presented.

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